

PATTERN RECOGNITION ON ORIENTED MATROIDS: TOPES AND CRITICAL COMMITTEES

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ABSTRACT. Let the sign components of the maximal covectors of a simple oriented matroid \mathcal{M} be represented by the real numbers -1 and 1 . Consider the vertex set $\mathfrak{V}(\mathbf{R})$ of a symmetric cycle \mathbf{R} of adjacent topes in the tope graph of \mathcal{M} as a subposet of the tope poset of \mathcal{M} . If B is the bottom element of the tope poset then B is equal to the unweighted sum of the members of the set $\mathbf{min} \mathfrak{V}(\mathbf{R})$ of minimal elements of the subposet $\mathfrak{V}(\mathbf{R})$; if B is the positive tope then the set $\mathbf{min} \mathfrak{V}(\mathbf{R})$ is a critical tope committee for the acyclic oriented matroid \mathcal{M} .

CONTENTS

1.	Introduction	1
2.	Topes and Critical Committees	3
	References	7

1. INTRODUCTION

Let $\mathcal{M} := (E_t, \mathcal{T})$ be an oriented matroid, of rank ≥ 2 , on the ground set $E_t := \{1, \dots, t\}$, with set of topes \mathcal{T} ; throughout we will suppose that it is *simple*, that is, it contains no loops, parallel or *antiparallel* elements. In this paper the sign components $-$ and $+$ of maximal covectors are replaced by the real numbers -1 and 1 , respectively. The topes $T := (T(1), \dots, T(t)) \in \mathcal{T}$ are interpreted as elements of the real Euclidean space \mathbb{R}^t of row vectors; if $T, T', T'' \in \mathbb{R}^t$ then $\langle T', T'' \rangle := \sum_{e=1}^t T'(e) \cdot T''(e)$, and $\|T\| := \sqrt{\langle T, T \rangle}$. We denote by $(\sigma(1), \dots, \sigma(t))$ the standard basis of \mathbb{R}^t , that is, $\sigma(i) := (0, \dots, \underset{\uparrow}{1}, \dots, 0)$, $1 \leq i \leq t$. The *positive tope* $T^{(+)}$ is the vector $(1, \dots, 1)$

$$= \sum_{i=1}^t \sigma(i).$$

If $e \in E_t$ then the corresponding *positive halfspace* \mathcal{T}_e^+ is the tope subset $\{T \in \mathcal{T} : T(e) = 1\}$.

If $T \in \mathcal{T}$ and $A \subseteq E_t$ then $-_A T$ denotes the vector obtained from T by *sign reversal* or *reorientation* on the set A : $(-_A T)(e) = -T(e)$ when $e \in A$,

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and $(-_A T)(e) = T(e)$ when $e \notin A$. The oriented matroid whose topes are obtained from the topes of \mathcal{M} by reorientation on the set A is denoted by $-_A \mathcal{M}$.

We denote by T^- the *negative part* $\{e \in E_t : T(e) = -1\}$ of the tope T ; the *positive part* T^+ of T is the set $\{e \in E_t : T(e) = 1\}$.

The vertices of the *tope graph* $\mathcal{T}(\mathcal{L}(\mathcal{M}))$ of the oriented matroid \mathcal{M} are its topes; a pair of topes $\{T', T''\} \subset \mathcal{T}$ is an edge of the graph $\mathcal{T}(\mathcal{L}(\mathcal{M}))$ if the topes T' and T'' are *adjacent*, that is, they cover some *subtope* in the *big face lattice* of \mathcal{M} . The *separation set* $\mathbf{S}(T', T'')$ of topes T' and T'' is defined by $\mathbf{S}(T', T'') := \{e \in E_t : T'(e) \neq T''(e)\}$. The *graph distance* $d(T', T'')$ between the topes T' and T'' is the cardinality of the separation set $\mathbf{S}(T', T'')$, see [1, Prop. 4.2.3], that is,

$$d(T', T'') = |\mathbf{S}(T', T'')| = t - \frac{1}{4} \|T'' + T'\|^2 = \frac{1}{4} \|T'' - T'\|^2.$$

If $B \in \mathcal{T}$ then we denote by $\mathcal{T}(\mathcal{L}(\mathcal{M}), B)$ the *tope poset* of \mathcal{M} based at the tope B ; by convention, we have $T' \preceq T''$ in $\mathcal{T}(\mathcal{L}(\mathcal{M}), B)$ iff $\mathbf{S}(B, T') \subseteq \mathbf{S}(B, T'')$. If $\mathcal{X} \subset \mathcal{T}(\mathcal{L}(\mathcal{M}), B)$ then $\mathbf{min} \mathcal{X}$ stands for the set of minimal elements of the subposet \mathcal{X} .

Let $\mathbf{m} := (R^0 := B \prec R^1 \prec \dots \prec R^{t-1} \prec R^t := -B)$ be a maximal chain in the tope poset $\mathcal{T}(\mathcal{L}(\mathcal{M}), B)$, and $-\mathbf{m} := \{-R : R \in \mathbf{m}\}$. The union $\mathfrak{V}(\mathbf{R}) := \mathbf{m} \cup -\mathbf{m}$ is the vertex set of the *symmetric cycle* $\mathbf{R} := (R^0 := B, R^1, \dots, R^{2t-1}, R^0)$ in the tope graph $\mathcal{T}(\mathcal{L}(\mathcal{M}))$; by convention, we have $R^{k+t} = -R^k$, $0 \leq k \leq t-1$. The subset of topes from $\mathfrak{V}(\mathbf{R})$ with inclusion-maximal positive parts is denoted by $\mathbf{max}^+(\mathfrak{V}(\mathbf{R}))$. If $A \subseteq E_t$ then $-_A \mathfrak{V}(\mathbf{R}) := \{-_A R : R \in \mathfrak{V}(\mathbf{R})\}$.

Let $(\mathfrak{l}_1, \dots, \mathfrak{l}_t) \in \mathbb{N}^t$ be the sequence defined by $\{\mathfrak{l}_i\} := \mathbf{S}(R^{i-1}, R^i)$; note that $\mathbf{m} - \{-B\} \subseteq \mathcal{T}_{\mathfrak{l}_t}^+$.

The chain $\mathbf{m} - \{-B\}$ is a basis of the space \mathbb{R}^t ; indeed, the square *sign matrix*

$$\mathbf{M} := \mathbf{M}(\mathbf{R}) := \begin{pmatrix} R^0 \\ R^1 \\ \vdots \\ R^{t-2} \\ R^{t-1} \end{pmatrix} \in \mathbb{R}^{t \times t}$$

is similar to the nonsingular matrix

$$\begin{pmatrix} 2 \cdot B(\mathfrak{l}_1) \cdot \sigma(\mathfrak{l}_1) \\ 2 \cdot B(\mathfrak{l}_2) \cdot \sigma(\mathfrak{l}_2) \\ \vdots \\ 2 \cdot B(\mathfrak{l}_{t-1}) \cdot \sigma(\mathfrak{l}_{t-1}) \\ B(\mathfrak{l}_t) \cdot \sigma(\mathfrak{l}_t) \end{pmatrix};$$

the absolute value of its determinant is 2^{t-1} .

The i th row $(\mathbf{M}^{-1})_i$, $1 \leq i \leq t$, of the inverse matrix \mathbf{M}^{-1} of \mathbf{M} is

$$(\mathbf{M}^{-1})_i = \begin{cases} \frac{1}{2} \cdot B(i) \cdot (\sigma(k) - \sigma(k+1)), & \text{if } i = \mathfrak{l}_k, k \neq t, \\ \frac{1}{2} \cdot B(i) \cdot (\sigma(1) + \sigma(t)), & \text{if } i = \mathfrak{l}_t. \end{cases}$$

Thus, if $T \in \mathcal{T}$ then we have $T = \mathbf{x}\mathbf{M}$ for some row vector

$$\mathbf{x} := (x_1, \dots, x_t) = T\mathbf{M}^{-1} \quad (1.1)$$

such that

$$\mathbf{x} \in \{-1, 0, 1\}^t .$$

A subset $\mathcal{K}^* \subset \mathcal{T}$ is called a *tope committee* for \mathcal{M} if

$$\sum_{T \in \mathcal{K}^*} T \geq \mathbf{T}^{(+)} ,$$

see [2, 3, 4, 5, 6]. The committee \mathcal{K}^* is called *minimal* if any its proper subset is not a committee for \mathcal{M} . If the sum of the members of the minimal tope committee \mathcal{K}^* is the positive tope,

$$\sum_{T \in \mathcal{K}^*} T = \mathbf{T}^{(+)} ,$$

then we say that the committee \mathcal{K}^* is *critical*.

Recall that if \mathbf{R} is a symmetric cycle in the tope graph $\mathcal{T}(\mathcal{L}(\mathcal{M}))$ of the oriented matroid \mathcal{M} , then for any tope $T \in \mathcal{T}$ there exists a unique inclusion-minimal subset $\mathbf{Q}(T, \mathbf{R}) \subset \mathfrak{V}(\mathbf{R})$ such that

$$T = \sum_{Q \in \mathbf{Q}(T, \mathbf{R})} Q ; \quad (1.2)$$

this set, of odd cardinality, of linearly independent elements of \mathbb{R}^t is

$$\mathbf{Q}(T, \mathbf{R}) = \{x_i \cdot R^{i-1} : x_i \neq 0\} ,$$

where the vector \mathbf{x} is defined by (1.1).

As a consequence, a tope subset $\mathcal{K}^* \subset \mathcal{T}$ is a committee for \mathcal{M} iff

$$\sum_{T \in \mathcal{K}^*} \sum_{Q \in \mathbf{Q}(T, \mathbf{R})} Q \geq \mathbf{T}^{(+)} .$$

In Section 2 we discuss relation (1.2) and describe the structure of the sets $\mathbf{Q}(T, \mathbf{R})$ in more detail.

2. TOPES AND CRITICAL COMMITTEES

For a tope T of the oriented matroid \mathcal{M} and for a symmetric cycle \mathbf{R} in its tope graph, we describe the corresponding inclusion-minimal subset $\mathbf{Q}(T, \mathbf{R}) \subset \mathfrak{V}(\mathbf{R})$ from expression (1.2) in terms of the tope poset of \mathcal{M} . Dual assertions could be made because the tope poset of \mathcal{M} is self-dual in the sense of [1, Prop. 4.2.15(ii)].

Theorem 2.1. *Let $\mathbf{R} := (R^0, R^1, \dots, R^{2t-1}, R^0)$ be a symmetric cycle in the tope graph $\mathcal{T}(\mathcal{L}(\mathcal{M}))$ of the oriented matroid \mathcal{M} . Pick a tope $B \in \mathcal{T}$ and consider the vertex set $\mathfrak{V}(\mathbf{R})$ of the cycle \mathbf{R} as a subposet of the tope poset $\mathcal{T}(\mathcal{L}(\mathcal{M}), B)$ based at B .*

The set $\mathbf{Q}(B, \mathbf{R})$ is the set $\min \mathfrak{V}(\mathbf{R})$ of minimal elements of the subposet $\mathfrak{V}(\mathbf{R}) \subset \mathcal{T}(\mathcal{L}(\mathcal{M}), B)$; therefore

$$B = \sum_{Q \in \min \mathfrak{V}(\mathbf{R})} Q. \quad (2.1)$$

Proof. If $B \in \mathfrak{V}(\mathbf{R})$, there is nothing to prove. Suppose that $B \notin \mathfrak{V}(\mathbf{R})$, and reorient the items of the poset $\mathcal{T}(\mathcal{L}(\mathcal{M}), B)$ on the negative part B^- of the tope B ; in other words, consider the tope poset $\mathcal{T}(\mathcal{L}_{-(B^-)}\mathcal{M}, T^{(+)})$ of the acyclic oriented matroid $_{-(B^-)}\mathcal{M}$ based at the positive tope $T^{(+)}$.

If O is a tope of the oriented matroid $_{-(B^-)}\mathcal{M}$, then the poset rank $d(T^{(+)}, O)$ of O in the tope poset $\mathcal{T}(\mathcal{L}_{-(B^-)}\mathcal{M}, T^{(+)})$ of $_{-(B^-)}\mathcal{M}$ is equal to the cardinality $|O^-| = |\mathbf{S}(T^{(+)}, O)|$ of the negative part of O .

A tope O of the oriented matroid $_{-(B^-)}\mathcal{M}$ belongs to the set $\max^+(_{-(B^-)}\mathfrak{V}(\mathbf{R}))$ iff for the 2-path (O', O, O'') , where $O' \neq O''$, in the symmetric cycle $(_{-(B^-)}R^0, _{-(B^-)}R^1, \dots, _{-(B^-)}R^{2t-1}, _{-(B^-)}R^0)$ we have $d(T^{(+)}, O') = d(T^{(+)}, O'') = d(T^{(+)}, O) + 1$. By [4, Prop. 5.6], the set $\max^+(_{-(B^-)}\mathfrak{V}(\mathbf{R})) = \min _{-(B^-)}\mathfrak{V}(\mathbf{R})$ of minimal elements of the subposet $_{-(B^-)}\mathfrak{V}(\mathbf{R}) \subset \mathcal{T}(\mathcal{L}_{-(B^-)}\mathcal{M}, T^{(+)})$ is the inclusion-minimal subset of topes with the property $\sum_{T \in \max^+(_{-(B^-)}\mathfrak{V}(\mathbf{R}))} T = T^{(+)}$, that is,

$$T^{(+)} = \sum_{T \in \min _{-(B^-)}\mathfrak{V}(\mathbf{R})} T. \quad (2.2)$$

This means that $\min _{-(B^-)}\mathfrak{V}(\mathbf{R})$ is a critical tope committee for the acyclic oriented matroid $_{-(B^-)}\mathcal{M}$. Relation (2.1) is equivalent to (2.2). \square

Corollary 2.2. *If \mathbf{R} is a symmetric cycle in the tope graph of the oriented matroid \mathcal{M} , then for any tope $T \in \mathcal{T}$ the corresponding set $\mathbf{Q}(T, \mathbf{R})$ is the set $_{-(T^-)}(\max^+(_{-(T^-)}\mathfrak{V}(\mathbf{R})))$; therefore*

$$T = \sum_{Q \in _{-(T^-)}(\max^+(_{-(T^-)}\mathfrak{V}(\mathbf{R})))} Q.$$

Example 2.3. *Consider the Hasse diagram, which is depicted in Figure 1(a), of the tope poset $\mathcal{T}(\mathcal{L}(\mathcal{M}), B)$ of a simple oriented matroid \mathcal{M} of rank 3, where $B := (-1, 1, 1, 1, 1)$. Fix the symmetric cycle $\mathbf{R} := ((1, -1, 1, 1, 1), (1, -1, 1, -1, 1), (1, -1, -1, -1, 1), (1, 1, -1, -1, 1), (-1, 1, -1, -1, 1), (-1, 1, -1, 1, -1), (-1, 1, 1, 1, -1), (-1, -1, 1, 1, -1), (1, -1, 1, 1, -1), (1, -1, 1, 1, 1))$ in the tope graph of \mathcal{M} .*

The Hasse diagram, borrowed from [1, Fig. 4.2.2], of the tope poset $\mathcal{T}(\mathcal{L}_{-\{1\}}\mathcal{M}, T^{(+)})$ of the acyclic oriented matroid $_{-\{1\}}\mathcal{M}$ is shown in Figure 1(b). We have $\min _{-(B^-)}\mathfrak{V}(\mathbf{R}) = \{(-1, -1, 1, 1, 1), (1, 1, -1, -1, 1),$

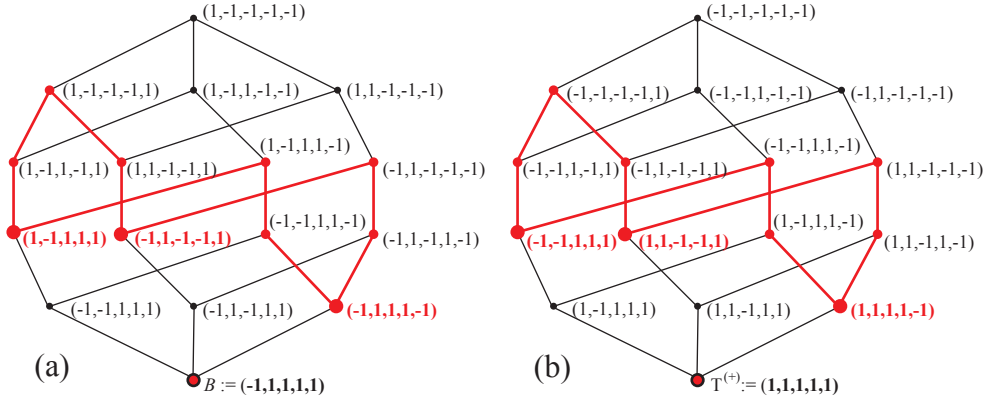


FIGURE 1. (a): The Hasse diagram of the tope poset $\mathcal{T}(\mathcal{L}(\mathcal{M}), B)$ based at the tope $B := (-1, 1, 1, 1, 1)$, and the **subposet** $\mathfrak{V}(\mathbf{R})$ which is the vertex set of a symmetric cycle \mathbf{R} in the tope graph. (b): The Hasse diagram of the tope poset $\mathcal{T}(\mathcal{L}_{(-\{1\}}\mathcal{M}), T^{(+)})$ based at the positive tope $T^{(+)} := (1, 1, 1, 1, 1)$.

$(1, 1, 1, 1, -1)\}$ and

$$T^{(+)} = \sum_{Q \in \min_{-(B^-)} \mathfrak{V}(\mathbf{R})} Q = (-1, -1, 1, 1, 1) + (1, 1, -1, -1, 1) + (1, 1, 1, 1, -1).$$

Similarly, $\min \mathfrak{V}(\mathbf{R}) = \{(1, -1, 1, 1, 1), (-1, 1, -1, -1, 1), (-1, 1, 1, 1, -1)\}$ and

$$\begin{aligned} B &= \sum_{Q \in \min \mathfrak{V}(\mathbf{R})} Q = (1, -1, 1, 1, 1) + (-1, 1, -1, -1, 1) + (-1, 1, 1, 1, -1) \\ &= (-1, 1, 1, 1, 1). \end{aligned}$$

The subposet of the poset $\mathcal{T}(\mathcal{L}(\mathcal{M}), B)$, depicted in Figure 2, is the vertex set of another symmetric cycle $\mathbf{R} := ((-1, -1, 1, 1, 1), (1, -1, 1, 1, 1), (1, -1, 1, -1, 1), (1, -1, -1, -1, 1), (1, 1, -1, -1, 1), (1, 1, -1, -1, -1), (-1, 1, -1, -1, -1), (-1, 1, -1, 1, -1), (-1, 1, 1, 1, -1), (-1, -1, 1, 1, -1), (-1, -1, 1, 1, 1))$ in the tope graph of \mathcal{M} . We have $\min \mathfrak{V}(\mathbf{R}) = \{(-1, -1, 1, 1, 1), (1, 1, -1, -1, 1), (-1, 1, 1, 1, -1)\}$ and

$$\begin{aligned} B &= \sum_{Q \in \min \mathfrak{V}(\mathbf{R})} Q = (-1, -1, 1, 1, 1) + (1, 1, -1, -1, 1) + (-1, 1, 1, 1, -1) \\ &= (-1, 1, 1, 1, 1). \end{aligned}$$

Let $\mathbf{R} := ((-1, 1, 1, 1, 1), (-1, -1, 1, 1, 1), (1, -1, 1, 1, 1), (1, -1, 1, -1, 1), (1, -1, -1, -1, 1), (1, -1, -1, -1, -1), (1, 1, -1, -1, -1), (-1, 1, -1, -1, -1), (-1, 1, -1, 1, -1), (-1, -1, 1, 1, -1), (-1, -1, 1, 1, 1))$

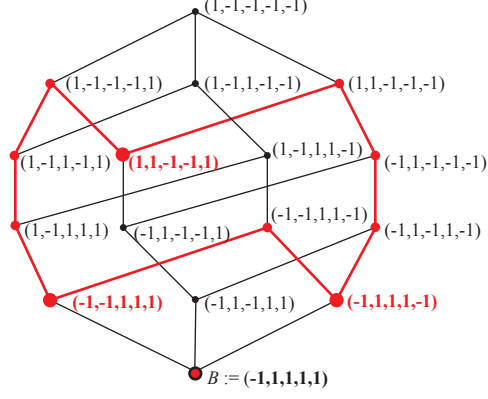


FIGURE 2. The Hasse diagram of the tope poset $\mathcal{T}(\mathcal{L}(\mathcal{M}), B)$ based at the tope $B := (-1, 1, 1, 1, 1)$, and the **subposet** $\mathfrak{V}(\mathbf{R})$ which is the vertex set of a symmetric cycle \mathbf{R} in the tope graph, cf. Figure 1(a).

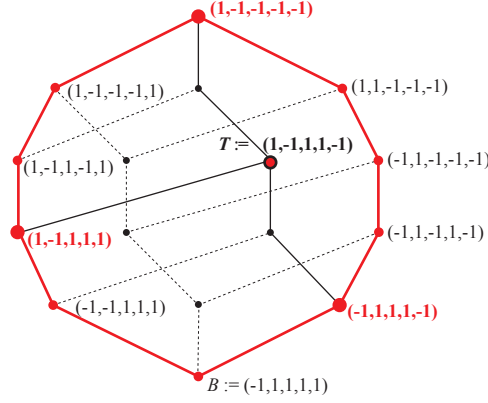


FIGURE 3. The Hasse diagram of the tope poset $\mathcal{T}(\mathcal{L}(\mathcal{M}), B)$ based at the tope $B := (-1, 1, 1, 1, 1)$, and the **subposet** $\mathfrak{V}(\mathbf{R})$ which is the vertex set of a symmetric cycle \mathbf{R} in the tope graph, cf. Figures 1(a) and 2.

$(-1, 1, -1, 1, -1), (-1, 1, 1, 1, -1), (-1, 1, 1, 1, 1))$ be one more symmetric cycle in the tope graph of the oriented matroid \mathcal{M} , see Figure 3. Pick the tope $T := (1, -1, 1, 1, -1)$ of \mathcal{M} . Since $_{-(T^-)}(\max^+(_{-(T^-)}\mathfrak{V}(\mathbf{R})))$

$= \{(1, -1, 1, 1, 1), (-1, 1, 1, 1, -1), (1, -1, -1, -1, -1)\}$, we have

$$\begin{aligned} T &:= (1, -1, 1, 1, -1) = \sum_{Q \in {}_{-(T-)}(\mathbf{max}^+({}_{-(T-)}\mathfrak{V}(\mathbf{R})))} Q \\ &= (1, -1, 1, 1, 1) + (-1, 1, 1, 1, -1) + (1, -1, -1, -1, -1) . \end{aligned}$$

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